Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. JANUARY, 1902. No. 1.

DETERMINATION OF ALL THE GROUPS OF ORDER 168.

By DR. G. A. MILLER, Leland Stanford Jr. University, Cal.

The first two composite numbers which are orders of simple groups are 60 and 168. Burnside determined the possible groups of the former of these orders in his Theory of Groups. The present paper is devoted to those of the latter order.

According to Sylow's theorem a group of order 168 contains either one or eight subgroups of order 7. If it contains 8 subgroups of order 7 it must permute them according to a transitive group of degree 8. As this transitive group involves substitutions of order 7 it must be doubly transitive. Hence such a group of order 168 must be isomorphic with a primitive group of degree 8 whose order is either 168 or 56. In the former case there is a simple isomorphism and the two possible groups are well known.* We proceed to prove that there is only one group of order 168 that contains 8 subgroups of order 7 and is isomorphic with the primitive group of degree 8 and order 56.

Since this primitive group contains an invariant subgroup of order 8 which involves 7 operators of order 2, such a group (G) contains an invariant subgroup (H) of order 24 which includes only operators of orders 2, 3, and 6. It contains only one subgroup of order 3. The two operators of order 3 must be invariant since the primitive group of degree 8 and order 56 does not include a subgroup of order 28. Hence H is composed of 7 operators of order 2 and 14 of order 6,

---

*Hoelder, *Mathematische Annalen*, Vol. 40, 1892, page 75; Cole, *Bulletin of the New York Mathematical Society*, Vol. 2, 1893, page 189.

besides the two operators of order 3 ; *i. e.* H is the direct product of its subgroups of orders 3 and 8. Moreover, G contains only one subgroup of order 24 since the group of degree 8 and order 56 contains only one subgroup of order 8.

In what follows we shall suppose that G is represented as a transitive group of degree 24.* Since H is regular and invariant G must be found in the holomorph of H. As the group of isomorphisms of H is the direct product of the groups of isomorphisms of its subgroups of orders 3 and 8,† all the subgroups of order 168 (including H) in the holomorph of H are conjugate ; *i. e. there is only one group of order* 168 *which contains* 8 *subgroups of order* 7 *and cannot be represented transitively on* 8 *letters.* Hence the total number of groups of order 168 which contain 8 subgroups of order 7 is three. In what follows we shall therefore assume that the groups under consideration contain only one subgroup of order 7.

1. GROUPS CONTAINING ONLY ONE SUBGROUP OF ORDER EIGHT.

If such a group contains only one subgroup of order 3, it is the direct product of its subgroups of orders 7, 8, and 3. Hence there are five such groups, corresponding to the five groups of order 8. If it contains four subgroups of order 3, its invariant subgroup of order 56 is the direct product of the group of order 7 and the quaternion group or the group of order 8 which involves no operator of order 4, since these are the only groups of order 8 whose groups of isomorphisms involve operators of order 3. The required groups must be in that part of the holomorphs of these groups of order 56 in which the operators of order 7 are invariant. Hence there are only two such groups, one for each of the given groups of order 8. They are the direct products of the two groups of order 24 which contain four subgroups of order 3 and only one subgroup of order 8.‡

When there are 7 subgroups of order 3 the group must be the direct product of the noncyclic group of order 21 and some subgroup of order 8, for each of its subgroups of order 24 contains only one subgroup of order 3. Hence, there are five such groups corresponding to the five groups of order 8. Finally, when there are 28 subgroups of order 3 the invariant subgroup of order 56 involves either the quaternion group or the group of order 7 which includes no operator of order 4. In each case there is just one group, since the operators of order 3 in the group of isomorphisms of these groups of order 8 are conjugate. Hence, there are just 14 groups of order 168 that contain only one subgroup of each of the orders 7 and 8.

2. GROUPS CONTAINING THREE SUBGROUPS OF ORDER EIGHT.

Each of these groups contains just three subgroups of order 56, which are the direct products of a subgroup of order 8 and the group of order 7. These conjugate subgroups of order 56 and the subgroups of order 8 are permuted

---

*Dyck, *Mathematische Annalen*, Vol. 22, 1883, page 91; *Bulletin of the American Mathematical Society*, Vol. 3, 1896, page 215.

†*Transactions of the American Mathematical Society*, Vol. 1, 1900, page 396.

‡*Quarterly Journal of Mathematics*, Vol. 28, 1896, page 274.

according to the symmetric group of order 6. Hence each of the required groups contains invariantly the direct product of the group of order 7 and a group of order 4. If such a group has either one or four subgroups of order 3 it must be the direct product of the group of order 7 and a group of order 24 which has three subgroups of order 8 and either one or four subgroups of order 3. Hence there are just eight such groups, seven of them involve only one subgroup of order 3, while the remaining one involves four such subgroups.

There cannot be either 7 or 28 subgroups of order 3, since such a group would include 56 operators that are commutative with an operator of order 7 and therefore it could not contain just three conjugate subgroups of order 56 as was stated above.

3. Groups Containing Seven Subgroups of Order Eight.

The subgroups of order 8 are permuted according to the transitive group of degree 7 and order 14 when the number of subgroups of order 3 is either one or four. When the number of these subgroups is either 7 or 28 the subgroups of order 8 are permuted according to the group of order 42 and degree 7. Each of the groups under consideration contains an invariant subgroup of order 4. In the former case there must be an invariant subgroup of order 84, which is the direct product of the group of order 7 and one of the three groups of order 12 which include only one group of order 4.

Three of the required groups may be obtained by forming the direct products of the group of order 14 and degree 7 multiplied by one of the three groups of order 12 which have just been mentioned. When the given invariant subgroup of order 84 is cyclic there are three additional groups.* In one of these the subgroups of order 8 are cyclic, in the second they are the quaternion group, and in the third each of them contains just two operators of order 4. Hence, there are just four groups of order 168 that contain 7 subgroups of order 8 and also an operator of order 84. When the invariant subgroup of order 84 is noncyclic and abelian there are two additional groups, whose subgroups of order 8 are the abelian group which has two invariants and the group which can be represented on four letters, respectively. Finally, when the given subgroup of order 84 is nonabelian there is no group in addition to the direct product mentioned above. The total number of groups of order 168, which contain an invariant subgroup of order 12 and just seven subgroups of order 8 is therefore 8.

By forming the direct products of the metacyclic group of degree 7 and the groups of order 4, we obtain two groups containing seven subgroups of order three. Each of the groups which contain seven such subgroups includes the direct product of the semimetacyclic group of degree 7 and one of the groups of order 4. When this group of order 4 is cyclic there are three additional groups, whose subgroups of order 8 are distinct. When this group of order 4 is noncyclic there are two more groups. Hence there are just 7 groups containing 7 subgroups of order 3.

***Transactions of the American Mathematical Society*, Vol. 2, 1901, page 264.

When there are 28 subgroups of order 3, they are permuted according to a simply isomorphic group of order 168, which contains a subgroup of order 84. As the operators of order 2 in this subgroup cannot transform a subgroup of order 3 into itself the required groups (or group) must include operators of order 2 which are not in the subgroup of order 84. Hence there is only one such group, and the total number of groups of order 168 which contain seven subgroups of order 8 is 16.

4. GROUPS CONTAINING TWENTY-ONE SUBGROUPS OF ORDER EIGHT.

Such a group contains three conjugate subgroups of order 56, which it permutes according to the symmetric group of order 6. Each of these subgroups contains 7 groups of order 8, which it permutes according to a group of order 14. Hence it contains just one invariant subgroup of order 4, while the entire group G contains only one invariant subgroup of order 28. Since the quotient group of G with respect to this invariant subgroup is the symmetric group of degree 3, G cannot contain more than four subgroups of order 3; for, if there were more than four such subgroups, the operators of order 7 would be permuted according to the cyclic group of order 6 and hence this group would be a quotient group.

The invariant subgroup of order 28 includes either seven or just one subgroup of order 4. In the former case they are permuted under G according to the group of order 14, as there cannot be an invariant subgroup of order 4 in this case, and hence there must be an invariant subgroup of order 12. Since there is also such an invariant subgroup in the latter case, each of the groups under consideration contains an invariant subgroup of order 12, and hence also the direct product of the group of order 7 and some group of order 12.

When this direct product is cyclic, there are four such groups, which may be distinguished by their subgroups of order 8. When the direct product is the other abelian group of order 84, there are three additional groups. In two of these the subgroups of order 8 are abelian while those of the third can be represented on four letters. Each of these seven groups contains only one abelian group of order 84. It remains to determine the groups which do not include an abelian subgroup of order 84.

When the subgroup of order 12 is the alternating group there is only one G. Hence there is only one group of order 168 that includes 21 subgroups of order 8 and 4 subgroups of order 3. As each of the remaining groups contains three conjugate subgroups of order 4, its invariant subgroup of order 28 includes seven subgroups of order 4. When the subgroup of order 12 includes six operators of order 4, there are four additional groups; two of them contain just four operators of order 4 in each subgroup of order 8, while the non-abelian subgroups of order 8 in the other two are distinct. The first two of these four groups may be distinguished by the fact that only the first of them contains operators of order 12.

The only case which remains is when the subgroup of order 12 includes 7 operators of order 2. In this case there are again four groups. Two of these contain distinct abelian subgroups of order 8, while the other two include the group of order 8 which can be represented on four letters. The latter may again be distinguished by the fact that only one of them contains operators of order 12. This concludes the examination of all the possible cases and proves that the total number of groups of order 168 that have 21 subgroups of order 8 is the same as the number of those containing 7 such subgroups, viz. 16. *The total number of groups of order* 168 *is therefore* 57. Only one of these is insolvable.

◆

ON LIMITS.

By DR. ARNOLD EMCH, University of Colorado.

In most of the elementary text-books of algebra and calculus the chapters on limits are treated in a rather indefinite manner. I suspect that the reason for this deficiency lies partly in the semi-philosophical nature of the subject, partly in the neglect of the authors to apply the results of the theory of functions to limitting processes.

As an example I mention the frequently occurring definition of the *limit of a variable.*

(*a*) "When according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the limit of the variable."

This definition restricts the variable to the members of a sequence as it will appear from the following proposition given by Harkness and Morley :*

I. "The numbers ξ_1, ξ_2, ξ_3, ... of a sequence are said to tend to the limit a, when to every positive number ε there corresponds a positive integer μ such that for ξ_μ and for all later members ξ_n of the sequence we have

$$| \xi_n - a | < \varepsilon."$$

a itself does not belong to the sequence. Terms in the sequence of all positivs integers 1, 2, 3, ...—, ∞ does not belong to the sequence, although ∞ is the limit of the sequence. Similarly, in the sequence

$$\tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \ldots.$$

0, which is its limit, does not belong to the sequence. Notice also that 0 and ∞, which shall be defined presently, are among the values which a limit can have.

An infinitely large quantity, or the symbol ∞ is defined as the indefinite quantity A which satisfies the inequality

---

***Introduction to Analytic Functions*, page 67.
See also Burkhardt: *Functionentheorie*, Vol. I., page 68.